MSYM amplitudes in the high-energy limit

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Bern-Dixon-Smirnov ansatz

an ansatz for MHV amplitudes in N=4 SUSY — Bern Dixon Smirnov 05

$$m_n = m_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \right]$$

$$= m_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$ $\lambda = g^2 N$ 't Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \, \hat{G}^{(l)} + \epsilon^2 \, f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a* — Korchemsky Radyuskin 86, Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ IR function, known through O(a^4) — Bern Dixon Smirnov 05, Cachazo Spradlin Volovich 07

Brief history of BDS ansatz

BDS ansatz checked through 3-loop 4-pt amplitude — Bern Dixon Smirnov 05

2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

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BDS ansatz shown to fail on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08

Hints of break-up also from strong-coupling expansion — Alday Maldacena 07

hexagon Wilson loop — Drummond Henn Korchemsky Sokatchev 07

multi-Regge limit — Bartels Lipatov Sabio-Vera 08

Colour decomposition of the tree-level *n*-gluon amplitude

$$\mathcal{M}_n^{(0)} = 2^{n/2}\, g^{n-2} \sum_{S_n/Z_n} \mathrm{tr}(T^{d_1} \cdots T^{d_n})\, m_n^{(0)}(1,\ldots,n)$$

$m_n^{(0)}(1,2,\ldots,n)$ colour-stripped amplitude

MHV amplitude $m_n^{(0)}(1,2,\ldots,n) = \dfrac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \cdots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$

Regge limit

4-pt amplitude $g_1\, g_2 \to g_3\, g_4$ in the Regge limit $s \gg -t$

$$m_4(1,2,3,4) = s\,[g\,C(p_2,p_3,\tau)]\,\frac{1}{t}\left(\frac{-s}{\tau}\right)^{\alpha(t)}[g\,C(p_1,p_4,\tau)]$$

$\alpha(t)$ Regge trajectory $C(p_2,p_3,\tau)$ coefficient function τ Regge-factorisation scale

$$\alpha(t) = \bar{g}^2\bar{\alpha}^{(1)}(t) + \bar{g}^4\bar{\alpha}^{(2)}(t) + \bar{g}^6\bar{\alpha}^{(3)}(t) + O(\bar{g}^8)$$

$$C(p_i,p_j,\tau) = C^{(0)}(p_i,p_j)\left(1 + \bar{g}^2\bar{C}^{(1)}(t,\tau) + \bar{g}^4\bar{C}^{(2)}(t,\tau) + \bar{g}^6\bar{C}^{(3)}(t,\tau) + \mathcal{O}(\bar{g}^8)\right)$$

$\bar{\alpha}^{(n)}(t)\,,\quad \bar{C}^{(n)}(t,\tau)$ are re-scaled loop coefficients

$$\bar{\alpha}^{(n)}(t) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon}\alpha^{(n)}\,,\quad \bar{C}^{(n)}(t,\tau) = \left(\frac{\mu^2}{-t}\right)^{n\epsilon}C^{(n)}(t,\tau)$$

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Because the Regge limit is exponential in the Regge trajectory, one can use (the logarithm of) the BDS ansatz to obtain the Regge trajectory to all loops

Naculich Schnitzer 07
Bartels Lipatov Sabio-Vera 08
Glover VDD 08

$$\alpha^{(l)}(\epsilon) = 2^{l-1} \alpha^{(1)}(l\epsilon) \left(\frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon\, \frac{l}{2}\, \hat{G}^{(l)} \right) + O(\epsilon) \qquad \alpha^{(1)}(\epsilon) = \frac{2}{\epsilon}$$

Regge factorisation of the 1-loop 4-pt amplitude

$$m_4^{(1)} = \bar{\alpha}^{(1)}(t)L + \ 2\bar{C}^{(1)}(t,\tau)$$

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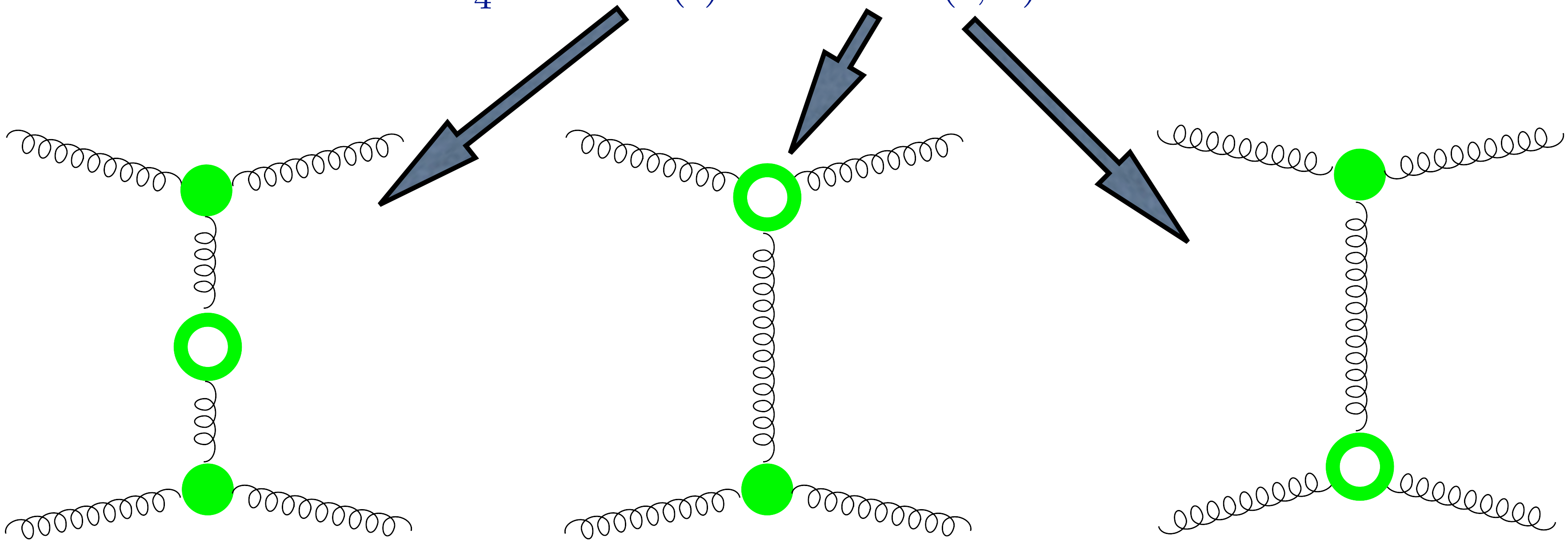

valid to all orders of ε

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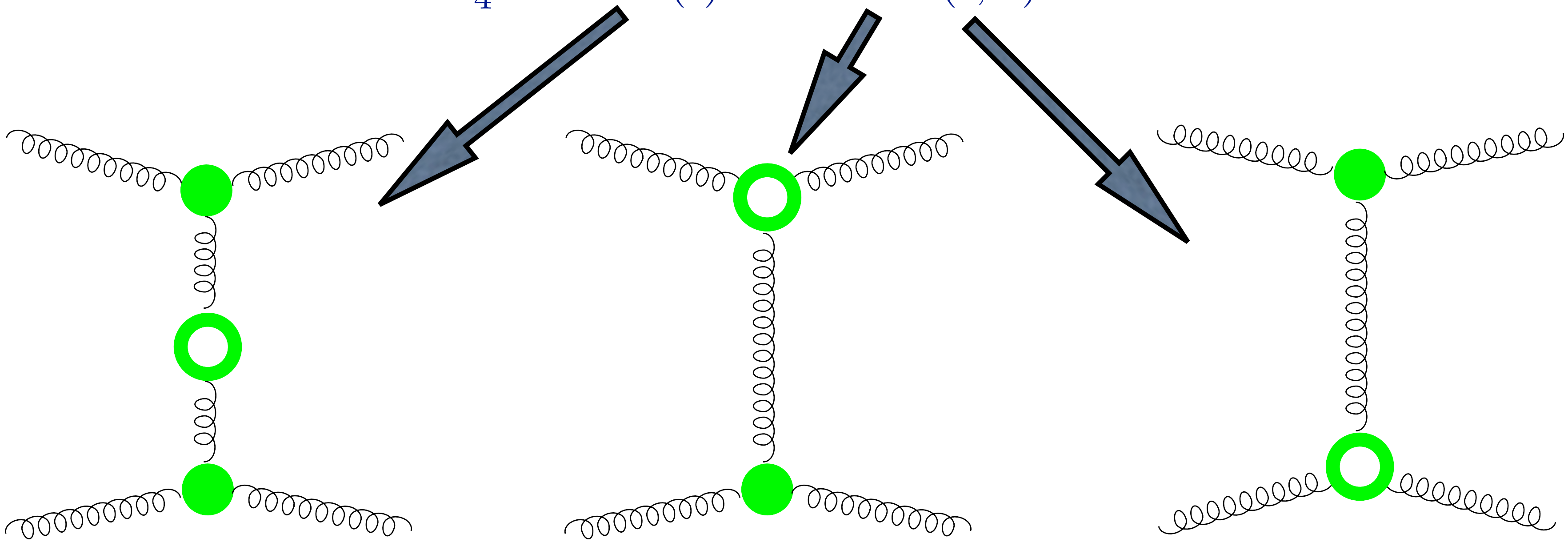


valid to all orders of ε

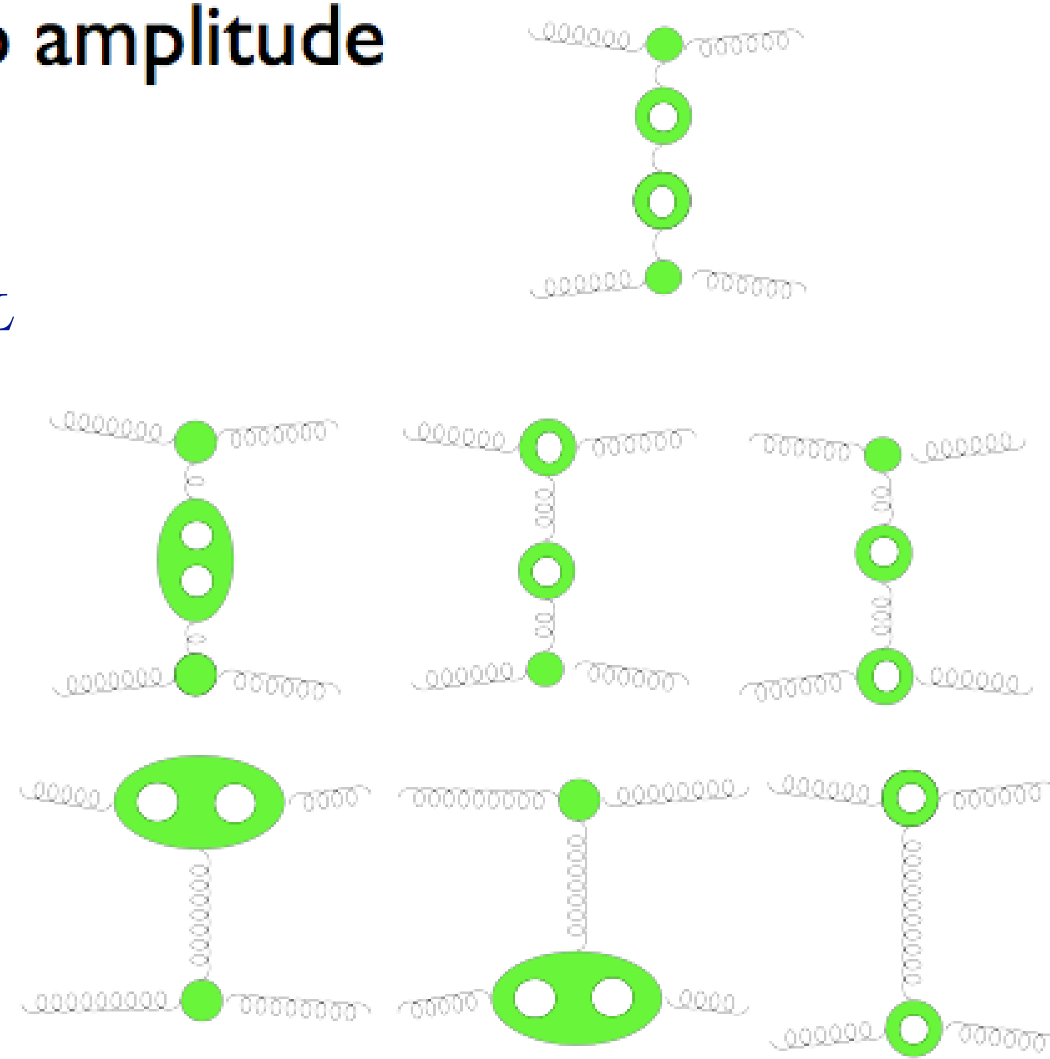
1-loop coefficient function

$$C^{(1)}(t,\tau) = \frac{\psi(1+\epsilon) - 2\psi(-\epsilon) + \psi(1)}{\epsilon} - \frac{1}{\epsilon}\ln\frac{-t}{\tau}$$

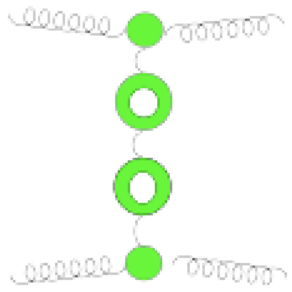
Factorisation of the 2-loop amplitude

$$
\begin{aligned}
m_4^{(2)} &= \frac{1}{2}\left(\bar{\alpha}^{(1)}(t)\right)^2 L^2 \\
&+ \left(\bar{\alpha}^{(2)}(t) + 2\,\bar{C}^{(1)}(t,\tau)\bar{\alpha}^{(1)}(t)\right) L \\
&+ 2\,\bar{C}^{(2)}(t,\tau) + \left(\bar{C}^{(1)}(t,\tau)\right)^2
\end{aligned}
$$

valid to all orders of ε



Factorisation of the 2-loop amplitude



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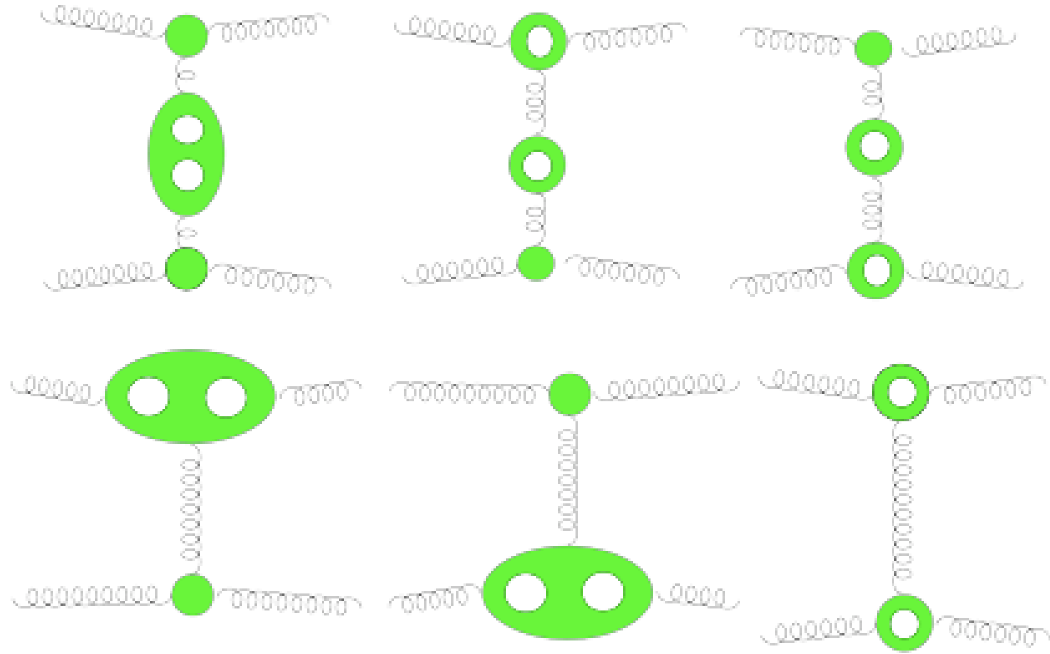

valid to all orders of ε

a more efficient way of writing it

$$
m_4^{(2)} = \frac{1}{2}\left(m_4^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + 2\,\bar{C}^{(2)}(t,\tau) - \left(\bar{C}^{(1)}(t,\tau)\right)^2
$$

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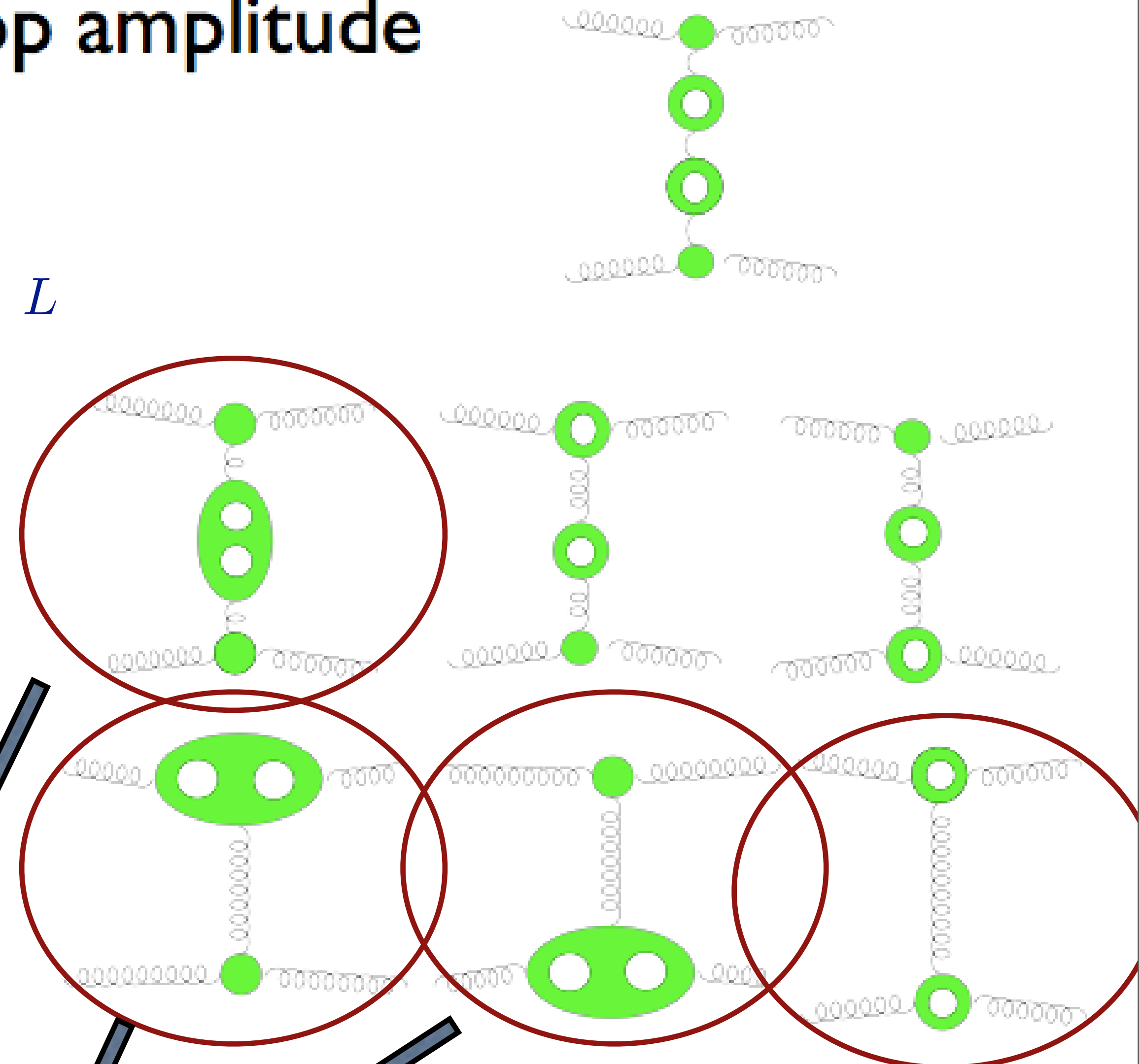

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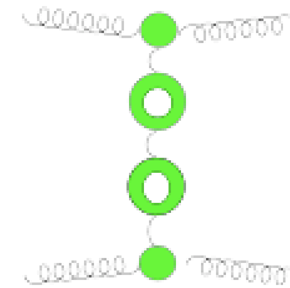

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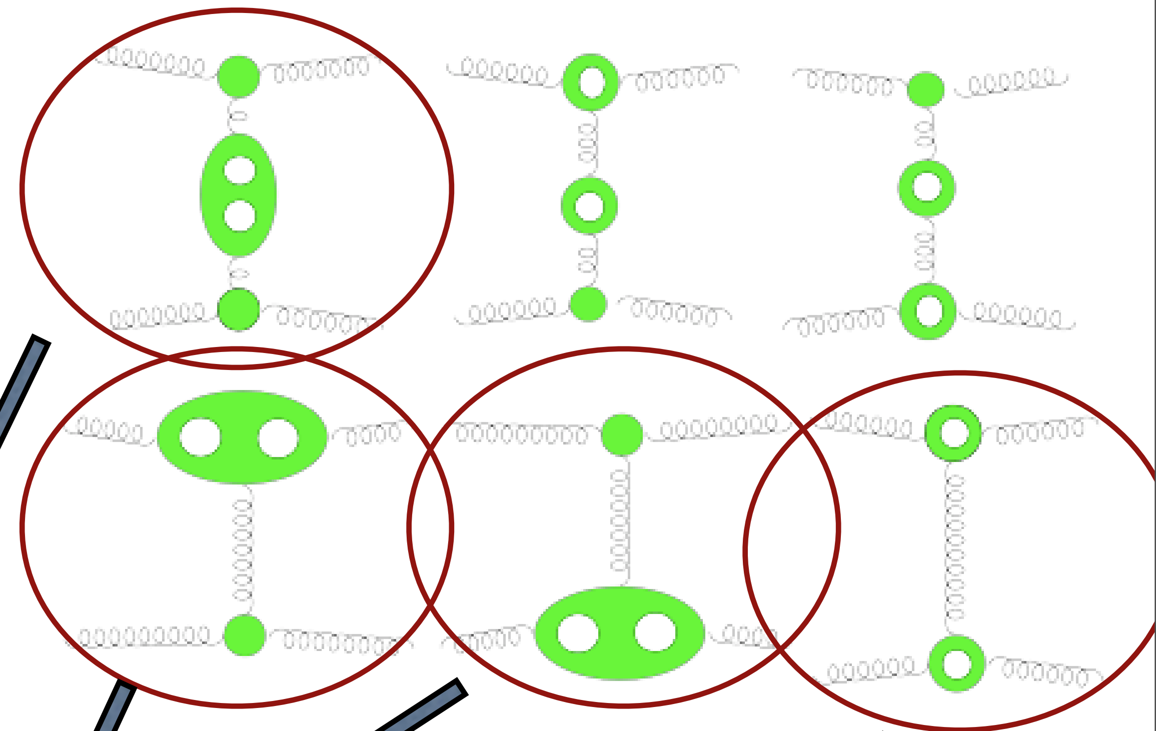

a more efficient way of writing it

$$
m_4^{(2)} = \frac{1}{2}\left(m_4^{(1)}\right)^2 + \bar{\alpha}^{(2)}(t)L + 2\,\bar{C}^{(2)}(t,\tau) - \left(\bar{C}^{(1)}(t,\tau)\right)^2
$$

where $m_4^{(1)}$ must be known through $\mathcal{O}(\epsilon^2)$

by direct calculation from
the 2-loop 4-pt amplitude $m_4^{(2)}$ to $O(\varepsilon^2)$
we get 2-loop trajectory

$$\alpha^{(2)} = -\frac{2\zeta_2}{\epsilon} - 2\zeta_3 - 8\zeta_4\epsilon + (36\zeta_2\zeta_3 + 82\zeta_5)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

2-loop coefficient function

$$\begin{aligned} C^{(2)}(t,\tau) &= \frac{1}{2}\left[C^{(1)}(t,\tau)\right]^2 + \frac{\zeta_2}{\epsilon^2} + \left(\zeta_3 + \zeta_2 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &+ \left(\zeta_3 \ln\frac{-t}{\tau} - 19\zeta_4\right) + \left(4\zeta_4 \ln\frac{-t}{\tau} - 2\zeta_2\zeta_3 - 39\zeta_5\right)\epsilon \\ &- \left(48\zeta_3^2 + \frac{1773}{8}\zeta_6 + (18\zeta_2\zeta_3 + 41\zeta_5)\ln\frac{-t}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

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where $C^{(1)}(t,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

A similar factorisation holds also for QCD amplitudes.
In that case, the 2-loop 4-parton amplitude $m_4^{(2)}$ yields the 2-loop trajectory

Fadin Fiore 95
Glover VDD 01

$$\alpha^{(2)} = C_A \left[\beta_0 \frac{1}{\epsilon^2} + K \frac{2}{\epsilon} + C_A \left(\frac{404}{27} - 2\zeta_3 \right) - \frac{56}{27} N_F \right] + \mathcal{O}(\epsilon)$$

maximal trascendentality
Kotikov Lipatov 02

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} N_F \qquad K = \left(\frac{67}{18} - \zeta_2 \right) C_A - \frac{5}{9} N_F$$

maximal trascendentality:

$\zeta_n, \ \ln^n, \ \epsilon^{-n}$ have weight *n* in trascendentality

MSYM amplitudes, and quantities derived from them, are homogeneous polynomials of maximal trascendentality

BDS ansatz and Regge limit

The BDS ansatz implies a 2-loop recursive formula
for the 2-loop n-pt amplitude $m_n^{(2)}$ (rescaled by the tree amplitude)

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + 4\, Const^{(2)} + \mathcal{O}(\epsilon)$$

Anastasiou Bern Dixon Kosower 03

valid for n = 4, 5

$$f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \qquad Const^{(2)} = -\frac{\zeta_2^2}{2}$$

(we use a different normalisation from BDS) $G(\epsilon) = \dfrac{e^{-\gamma\epsilon}\,\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)} = 1 + \mathcal{O}(\epsilon^2)$

from the recursive formula and Regge factorisation
we obtain recursive formulae for the Regge trajectory and the coefficient function

$$\alpha^{(2)}(\epsilon) = 2\, f^{(2)}(\epsilon)\,\alpha^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

$$C^{(2)}(t,\tau,\epsilon) = \frac{1}{2}\left[C^{(1)}(t,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\, C^{(1)}(t,\tau,2\epsilon) + 2\, Const^{(2)} + \mathcal{O}(\epsilon)$$

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where $C^{(1)}(t,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

the recursive formulae for $n = 4$ implied by
the BDS ansatz and by Regge factorisation differ in that
BDS: valid for arbitrary kinematics, but to $\mathcal{O}(\epsilon^0)$
Regge: valid to all orders of ε, but only in the Regge kinematics.
They overlap and agree in the Regge kinematics to $\mathcal{O}(\epsilon^0)$

Regge factorisation at 3 loops

$$m_4^{(3)} = m_4^{(2)} m_4^{(1)} - \frac{1}{3}\left(m_4^{(1)}\right)^3 + \bar{\alpha}^{(3)}(t)L + 2\,\bar{C}^{(3)}(t,\tau) - 2\,\bar{C}^{(2)}(t,\tau)\,\bar{C}^{(1)}(t,\tau) + \frac{2}{3}\left(\bar{C}^{(1)}(t,\tau)\right)^3$$

with 3-loop trajectory

$$\alpha^{(3)} = \frac{44\zeta_4}{3\epsilon} + \frac{40}{3}\zeta_2\zeta_3 + 16\zeta_5 + \mathcal{O}(\epsilon)$$

3-loop coefficient function

$$\begin{aligned} C^{(3)}(t,\tau) &= C^{(2)}(t,\tau)\,C^{(1)}(t,\tau) - \frac{1}{3}\left[C^{(1)}(t,\tau)\right]^3 \\ &- \frac{44}{9}\frac{\zeta_4}{\epsilon^2} - \left(\frac{40}{9}\zeta_2\zeta_3 + \frac{16}{3}\zeta_5 + \frac{22}{3}\zeta_4 \ln\frac{-t}{\tau}\right)\frac{1}{\epsilon} \\ &+ \frac{3982}{27}\zeta_6 - \frac{68}{9}\zeta_3^2 - \left(8\zeta_5 + \frac{20}{3}\zeta_2\zeta_3\right)\ln\frac{-t}{\tau} + \mathcal{O}(\epsilon) \end{aligned}$$

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where $C^{(1)}(t,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^4)$

$C^{(2)}(t,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

BDS ansatz and 3-loop Regge factorisation

from BDS's recursive formula for the 3-loop 4-point amplitude and Regge factorisation, we get recursive formulae for the 3-loop Regge trajectory and coefficient function

$$\alpha^{(3)}(\epsilon) = 4\, f^{(3)}(\epsilon)\, \alpha^{(1)}(3\epsilon) + \mathcal{O}(\epsilon)$$

$$\begin{aligned} C^{(3)}(t,\tau,\epsilon) &= C^{(2)}(t,\tau,\epsilon)\, C^{(1)}(t,\tau,\epsilon) - \frac{1}{3}\left[C^{(1)}(t,\tau,\epsilon)\right]^3 \\ &+ \frac{4\, G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\, C^{(1)}(t,\tau,3\epsilon) + 4\, Const^{(3)} + \mathcal{O}(\epsilon) \end{aligned}$$

with $$f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + (6\zeta_5 + 5\zeta_2\zeta_3)\epsilon + (c_1\zeta_6 + c_2\zeta_3^2)\epsilon^2$$

$$Const^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2$$

with c_1 and c_2 known constants (which drop out of the recursive formula above)

To $\mathcal{O}(\epsilon^0)$, the BDS recursive formula above is in agreement with the Regge recursive formula of the previous slide

Regge factorisation is valid also for amplitudes with 5 or more points
in generalised Regge limits.
The general strategy is to use the modular form
of the amplitudes dictated by high-energy factorisation,
to obtain information on n-point amplitudes in terms of building blocks derived
from m-point amplitudes, with $m < n$

Regge factorisation of the 5-pt amplitude

5-pt amplitude $g_1 g_2 \to g_3 g_4 g_5$ in the multi-Regge limit $s \gg s_1, s_2 \gg -t_1, -t_2$

$$m_5 = s\,[g\,C(p_2,p_3,\tau)]\;\frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa,\tau)]\;\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_5,\tau)]$$

Lipatov vertex

Regge factorisation of the 5-pt amplitude

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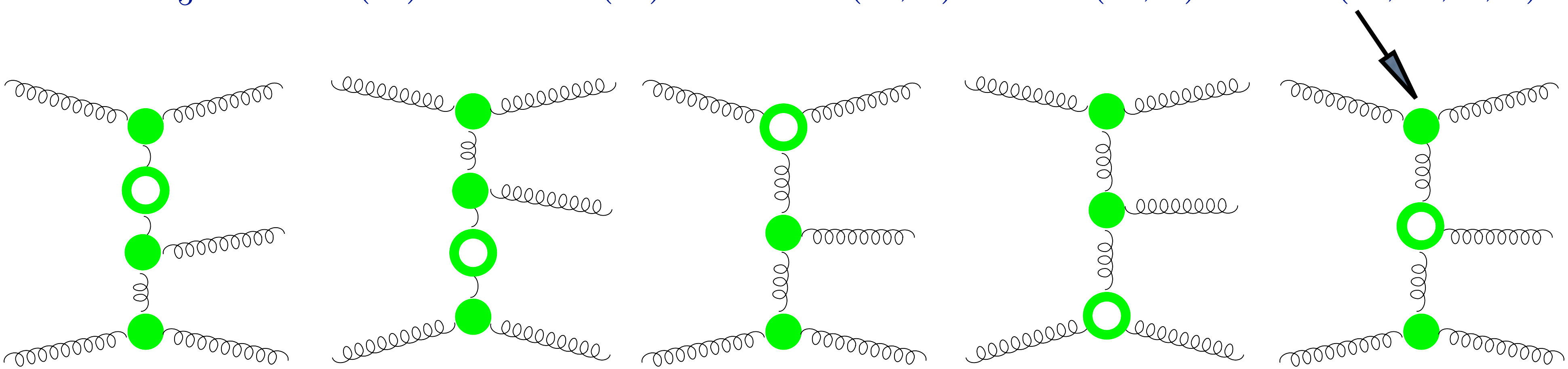
Lipatov vertex

1 loop

$$m_5^{(1)} = \bar{\alpha}^{(1)}(t_1)L_1 + \bar{\alpha}^{(1)}(t_2)L_2 + \bar{C}^{(1)}(t_1,\tau) + \bar{C}^{(1)}(t_2,\tau) + \bar{V}^{(1)}(t_1,t_2,\kappa,\tau)$$



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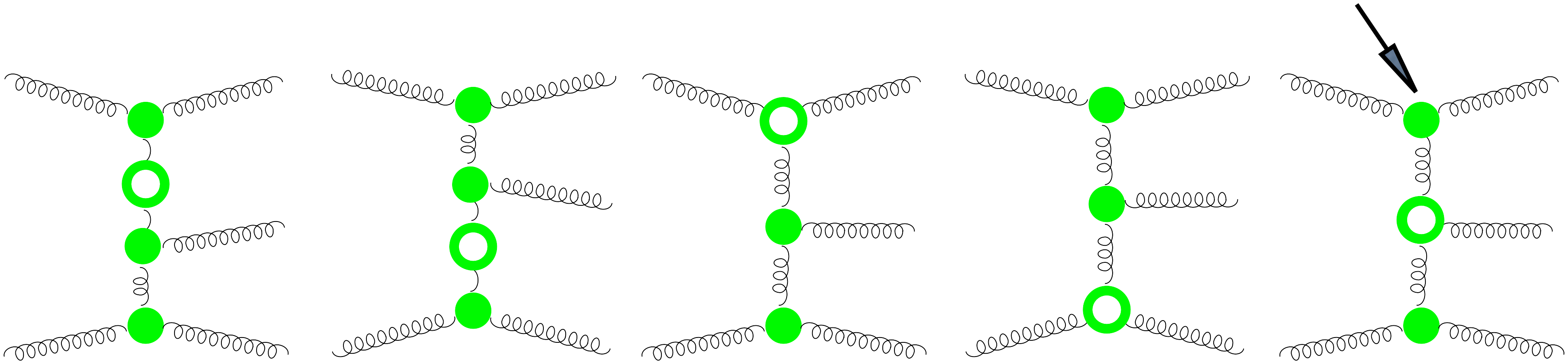
$$m_5 = s\,[g\,C(p_2,p_3,\tau)]\ \frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} [g\,V(q_2,q_1,\kappa,\tau)]\ \frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} [g\,C(p_1,p_5,\tau)]$$

Lipatov vertex

1 loop

$$m_5^{(1)} = \bar\alpha^{(1)}(t_1)L_1 + \bar\alpha^{(1)}(t_2)L_2 + \bar C^{(1)}(t_1,\tau) + \bar C^{(1)}(t_2,\tau) + \bar V^{(1)}(t_1,t_2,\kappa,\tau)$$



2 loops

$$\begin{aligned} m_5^{(2)} &= \frac{1}{2}\left(m_5^{(1)}\right)^2 + \bar\alpha^{(2)}(t_1)L_1 + \bar\alpha^{(2)}(t_2)L_2 \\ &+ \bar C^{(2)}(t_1,\tau) + \bar V^{(2)}(t_1,t_2,\kappa,\tau) + \bar C^{(2)}(t_2,\tau) \\ &- \frac{1}{2}\left(\bar C^{(1)}(t_1,\tau)\right)^2 - \frac{1}{2}\left(\bar V^{(1)}(t_1,t_2,\kappa,\tau)\right)^2 - \frac{1}{2}\left(\bar C^{(1)}(t_2,\tau)\right)^2 \end{aligned}$$

where $m_5^{(1)}$ must be known through $\mathcal{O}(\epsilon^2)$

BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and the Regge 2-loop recursive formula for the 5-pt amplitude $m_5^{(2)}$ and the recursive formulae for the Regge trajectory and the coefficient functions, one obtains a 2-loop recursive formula for the Lipatov vertex

$$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon) = \frac{1}{2}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\,V^{(1)}(t_1,t_2,\kappa,\tau,2\epsilon) + \mathcal{O}(\epsilon)$$

Duhr Glover VDD 08

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

BDS ansatz and Regge limit for the 5-pt amplitude

Using the BDS and the Regge 2-loop recursive formula for the 5-pt amplitude $m_5^{(2)}$ and the recursive formulae for the Regge trajectory and the coefficient functions, one obtains a 2-loop recursive formula for the Lipatov vertex

$$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon) = \frac{1}{2}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^2 + \frac{2\,G^2(\epsilon)}{G(2\epsilon)} f^{(2)}(\epsilon)\,V^{(1)}(t_1,t_2,\kappa,\tau,2\epsilon) + \mathcal{O}(\epsilon)$$

Duhr Glover VDD 08

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^2)$

Similarly, at 3 loops

$$\begin{aligned} V^{(3)}(t_1,t_2,\kappa,\tau,\epsilon) &= V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon) - \frac{1}{3}\left[V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)\right]^3 \\ &+ \frac{4\,G^3(\epsilon)}{G(3\epsilon)} f^{(3)}(\epsilon)\,V^{(1)}(t_1,t_2,\kappa,\tau,3\epsilon) + \mathcal{O}(\epsilon) \end{aligned}$$

where $V^{(1)}(t_1,t_2,\kappa,\tau,\epsilon)$ must be known through $\mathcal{O}(\epsilon^4)$

$V^{(2)}(t_1,t_2,\kappa,\tau,\epsilon)$ $\mathcal{O}(\epsilon^2)$

Regge factorisation of the 6-pt amplitude

6-pt amplitude $g_1 g_2 \to g_3 g_4 g_5 g_6$

in the multi-Regge limit $y_3 \gg y_4 \gg y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|$

$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$$

$$m_6 = s\,[g\,C(p_2, p_3, \tau)]\,\frac{1}{t_3}\left(\frac{-s_3}{\tau}\right)^{\alpha(t_3)}[g\,V(q_2, q_3, \kappa_2, \tau)]$$

$$\times \quad \frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_1, q_2, \kappa_1, \tau)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1, p_6, \tau)]$$

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$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$$

$$m_6 = s\, [g\, C(p_2, p_3, \tau)]\, \frac{1}{t_3} \left(\frac{-s_3}{\tau}\right)^{\alpha(t_3)} [g\, V(q_2, q_3, \kappa_2, \tau)]$$
$$\times \quad \frac{1}{t_2} \left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)} [g\, V(q_1, q_2, \kappa_1, \tau)]\, \frac{1}{t_1} \left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)} [g\, C(p_1, p_6, \tau)]$$

no new vertices or coefficient functions appear, wrt *n* = 5

The *l*-loop 6-pt amplitude can then be assembled using the *l*-loop trajectories, vertices and coefficient functions, which can be determined through the *l*-loop 4-pt and 5-pt amplitudes

Regge factorisation of the 6-pt amplitude

6-pt amplitude $g_1 g_2 \to g_3 g_4 g_5 g_6$

in the multi-Regge limit $y_3 \gg y_4 \gg y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq | p_{5\perp}| \simeq | p_{6\perp}|$

$$s \gg s_1, s_2, s_3 \gg -t_1, -t_2, -t_3$$

$$m_6 = s \left[g\, C(p_2, p_3, \tau) \right] \frac{1}{t_3} \left(\frac{-s_3}{\tau} \right)^{\alpha(t_3)} \left[g\, V(q_2, q_3, \kappa_2, \tau) \right]$$
$$\times \quad \frac{1}{t_2} \left(\frac{-s_2}{\tau} \right)^{\alpha(t_2)} \left[g\, V(q_1, q_2, \kappa_1, \tau) \right] \frac{1}{t_1} \left(\frac{-s_1}{\tau} \right)^{\alpha(t_1)} \left[g\, C(p_1, p_6, \tau) \right]$$

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The *l*-loop 6-pt amplitude can then be assembled using the *l*-loop trajectories, vertices and coefficient functions, which can be determined through the *l*-loop 4-pt and 5-pt amplitudes

Thus, also the *l*-loop BDS iterative formula for *n* = 6 will be fulfilled

⇒ the multi-Regge limit is not able to detect the BDS-ansatz violation for *n* = 6

Regge factorisation of the *n*-pt amplitude

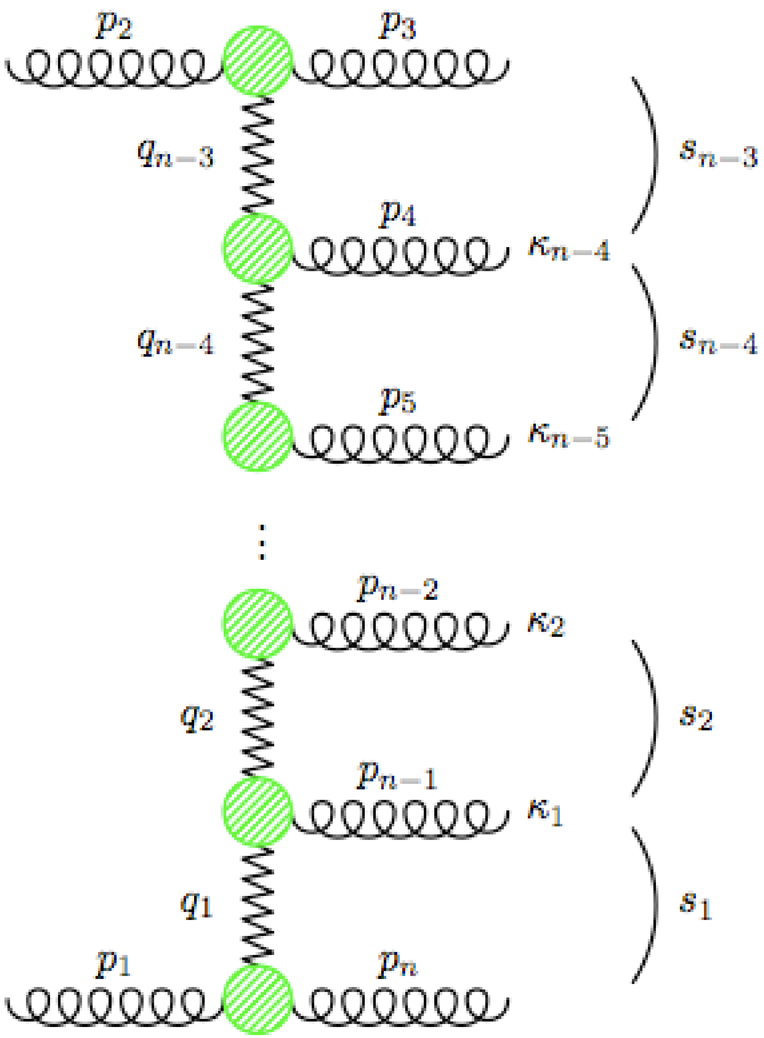
$$m_n(1,2,\ldots,n) = s\,[g\,C(p_2,p_3)]\,\frac{1}{t_{n-3}}\left(\frac{-s_{n-3}}{\tau}\right)^{\alpha(t_{n-3})}[g\,V(q_{n-3},q_{n-4},\kappa_{n-4})]$$

$$\cdots\times\ \frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}[g\,V(q_2,q_1,\kappa_1)]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}[g\,C(p_1,p_n)]$$

n-pt amplitude in the multi-Regge limit

$$y_3 \gg y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| ... \simeq |p_{n\perp}|$$

$$s \gg s_1, s_2, \ldots, s_{n-3} \gg -t_1, -t_2 \ldots, -t_{n-3}$$



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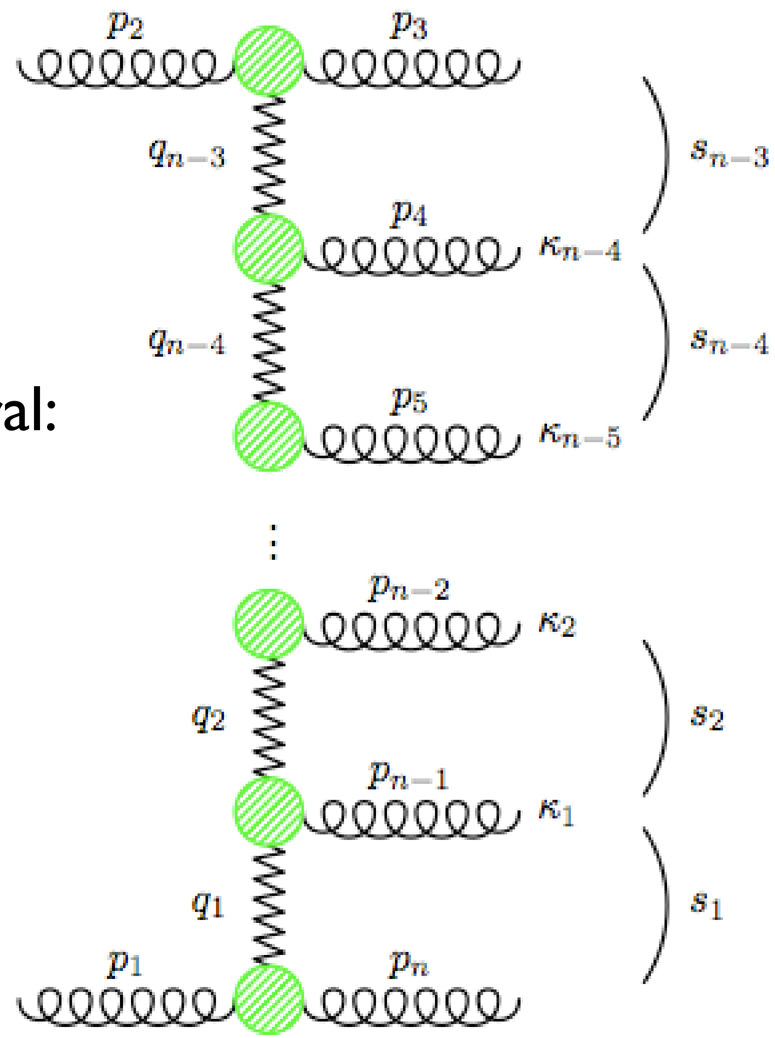


What we said for $n = 6$ can be repeated in general: the *l*-loop *n*-pt amplitude can be assembled using the *l*-loop trajectories, vertices and coefficient functions, determined through the *l*-loop 4-pt and 5-pt amplitudes

⟹ no violation of the BDS ansatz can be found in the multi-Regge limit

To have a chance to detect the violation of the BDS ansatz for the 2-loop 6-pt amplitude, that we see in arbitrary kinematics, we must relax the strong-ordering constraints of the multi-Regge kinematics

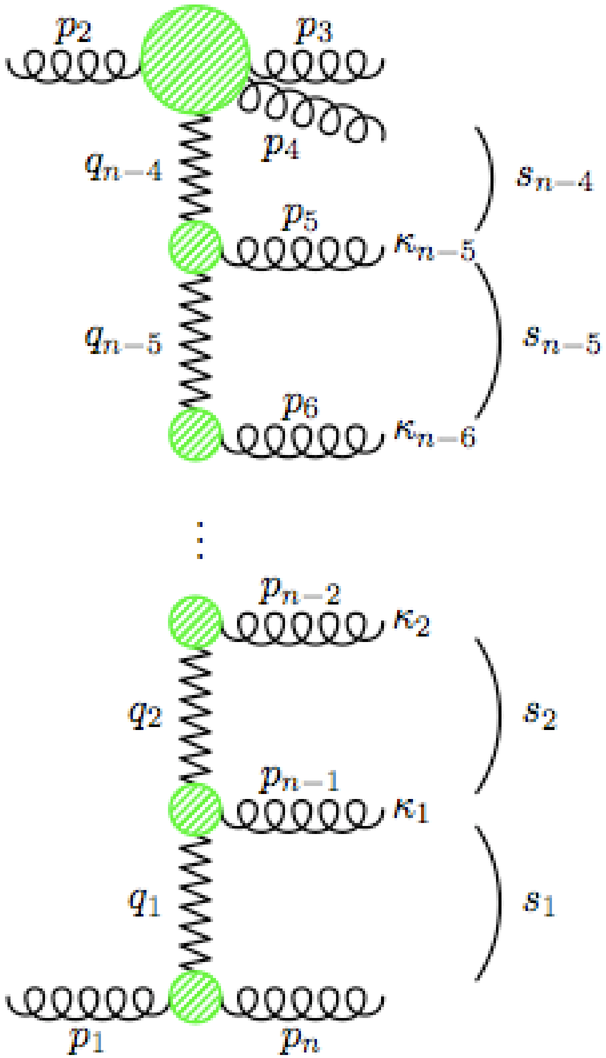
n-pt amplitude in quasi-multi-Regge kinematics

$$m_n(1,2,\ldots,n) = s\left[g^2\,A(p_2,p_3,p_4)\right]\,\frac{1}{t_{n-4}}\left(\frac{-s_{n-4}}{\tau}\right)^{\alpha(t_{n-4})}\left[g\,V(q_{n-4},q_{n-5},\kappa_{n-5})\right]$$

$$\cdots\times\ \frac{1}{t_2}\left(\frac{-s_2}{\tau}\right)^{\alpha(t_2)}\left[g\,V(q_2,q_1,\kappa_1)\right]\,\frac{1}{t_1}\left(\frac{-s_1}{\tau}\right)^{\alpha(t_1)}\left[g\,C(p_1,p_n)\right]$$

quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$



n-pt amplitude in quasi-multi-Regge kinematics

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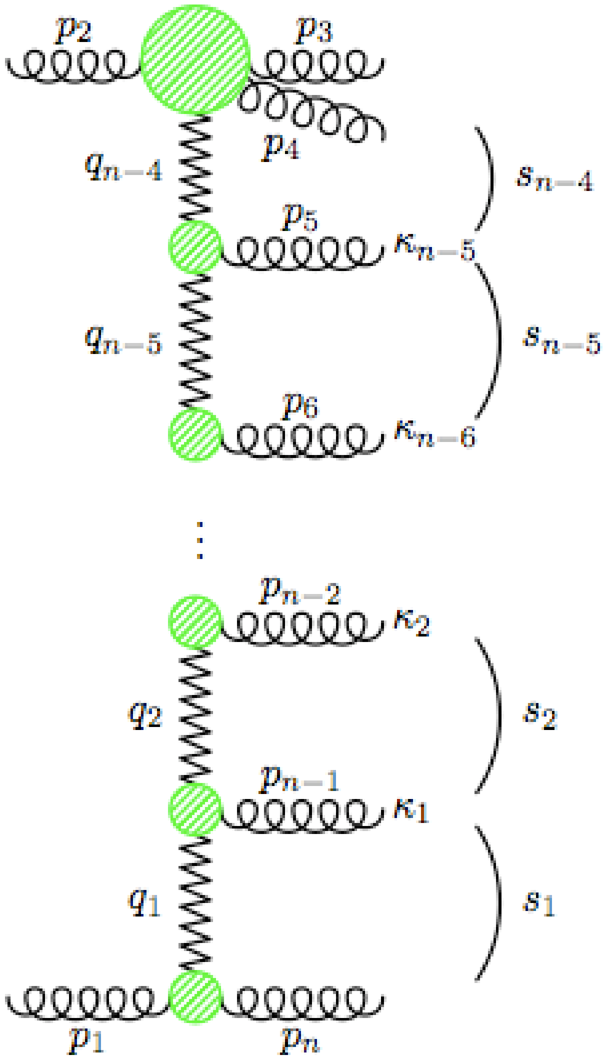
quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \ldots \simeq |p_{n\perp}|$$

A new coefficient function $A(p_2,p_3,p_4,\tau)$
occurs already at n = 5,
for which the BDS ansatz is fulfilled.
Because no new coefficient functions
appear for $n \geq 6$, a violation of the
BDS ansatz cannot be found even in this case



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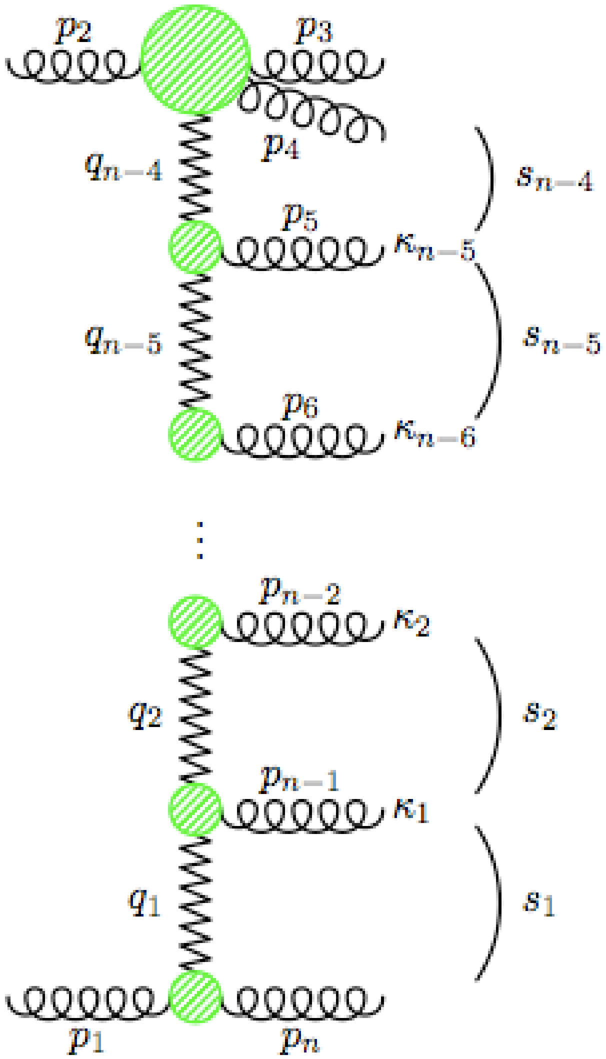
The same can be said for the
quasi-multi-Regge kinematics

$$y_3 \simeq y_4 \gg \cdots \gg y_{n-1} \simeq y_n; \qquad |p_{3\perp}| \simeq |p_{4\perp}|\ldots \simeq |p_{n\perp}|$$



More general quasi-multi-Regge kinematics

A necessary condition to see a violation of the BDS ansatz for the 2-loop 6-pt amplitude, is to go to a quasi-multi-Regge kinematics for which new coefficient functions appear for $n \geq 6$

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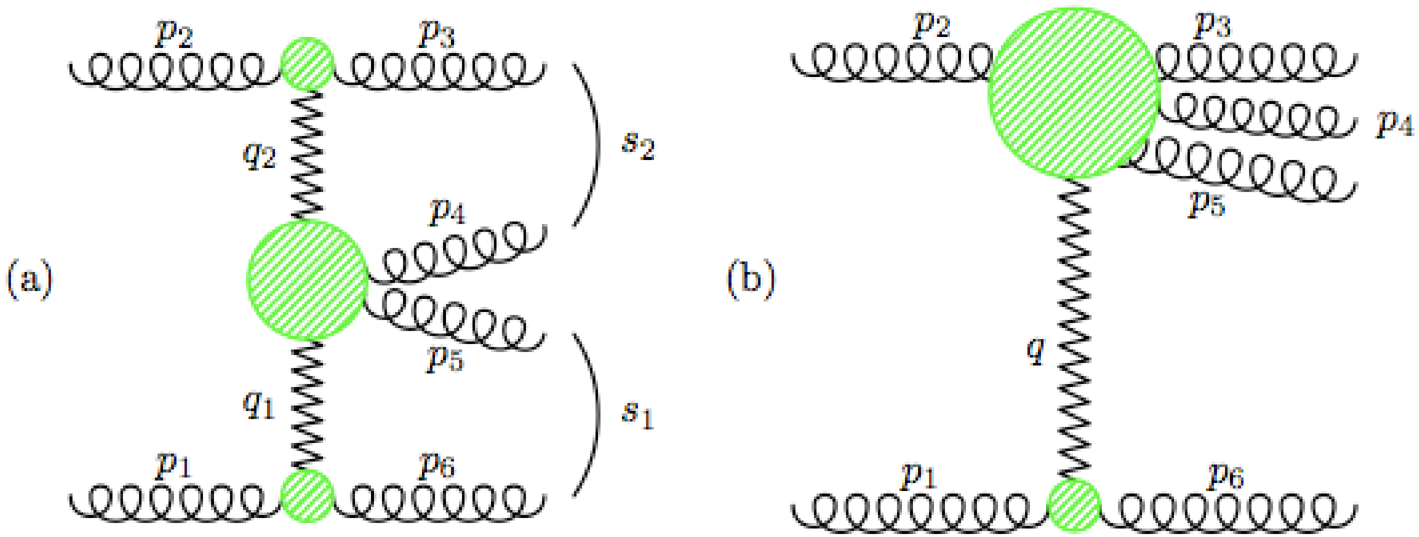

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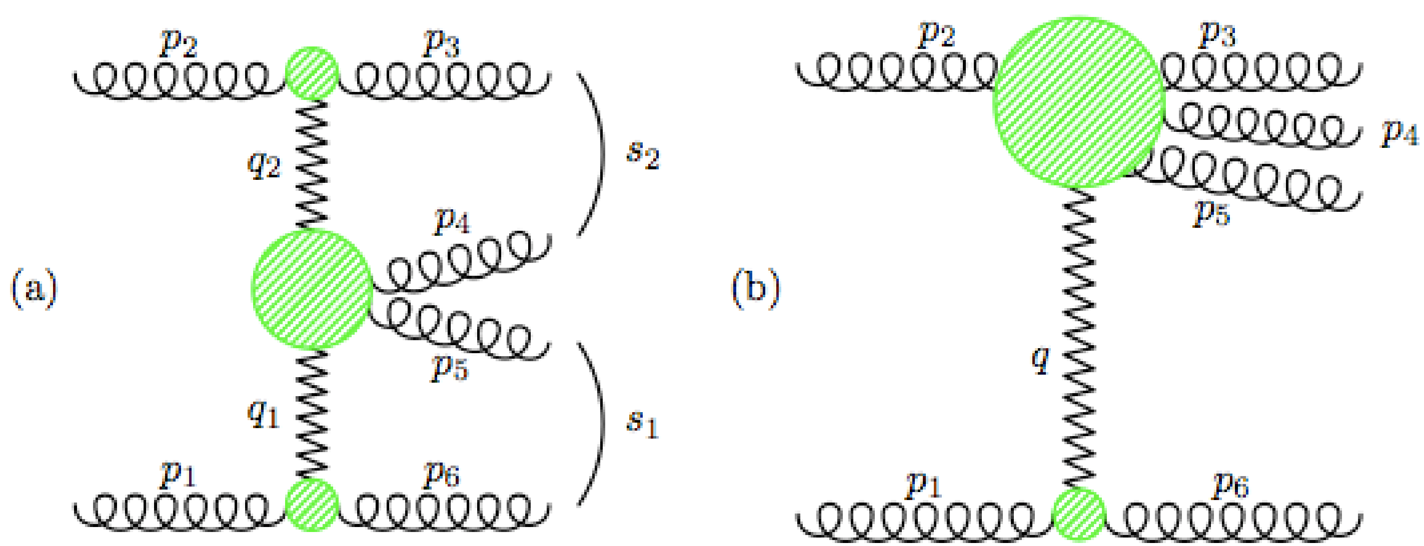

it remains to be seen if they harbour a violation of the BDS ansatz

Conclusions

- Using the Regge factorisation of the *l*-loop *n*-pt colour-stripped amplitude, we can build that amplitude in the multi-Regge kinematics in terms of a set of *l*-loop coefficient functions and vertices

- *the l*-loop *n*-pt colour-stripped amplitude thus built fulfils the BDS ansatz, thus any ansatz violation must be searched in less constraining (quasi-multi-Regge ?) kinematics